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Multiplicity of radially symmetric solutions for a p-harmonic equation in $\mathbb{R}^N$

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Abstract

This paper is concerned with the multiplicity of radially symmetric positive solutions of the Dirichlet boundary value problem for the following N-dimensional p-harmonic equation of the form

$$\Delta(|\Delta u|^{p-2}\Delta u) = \lambda g(x)f(u), \quad x \in B_1,$$

where B_1 is a unit ball in $\mathbb{R}^N$ ($N \geq 3$). We apply the fixed point index theory and the upper and lower solutions method to investigate the multiplicity of radially symmetric positive solutions. We find that there exists a threshold $\lambda^* < +\infty$ such that if $\lambda > \lambda^*$, the problem has no radially symmetric positive solution; while if $0 < \lambda \leq \lambda^*$, the problem admits at least one radially symmetric positive solution. Especially, there exist at least two radially symmetric positive solutions for $0 < \lambda < \lambda^*$.

1 Introduction

This paper is devoted to the study of radially symmetric positive solutions of the following boundary value problem for the N-dimensional quasilinear biharmonic equation:

$$\begin{aligned} &\Delta\big(|\Delta u|^{p-2}\Delta u\big) = \lambda g(x)f(u), \quad x \in B_1, \\ &u = 0, \quad x \in \partial B_1, \\ &\Delta u = 0, \quad x \in \partial B_1, \end{aligned} \tag{P}$$

where $B_1 = \{x \in \mathbb{R}^N \mid |x| < 1\}$, $N \geq 3$, $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, $p > 1$ is a constant, and $\lambda > 0$ is a positive parameter. In order to discuss the radially symmetric solution, we assume that $g(x)$ is radially symmetric, namely, $g(x) = g(|x|)$. Let $u(t) \triangleq u(|x|)$ with $t = |x|$ be a radially symmetric solution, then direct calculations show that

$$\mathscr{L}\big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big) = \lambda g(t)f(u), \quad t = |x|, 0 < t < 1, \tag{1.1}$$

with the boundary value condition

$$u'(0) = u(1) = \big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big)'|_{t=0} = \big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big)_{t=1} = 0, \tag{1.2}$$

where $\mathscr{L}$ denotes the polar form of the two-dimensional Laplacian Δ, *i.e.*,

$$\mathscr{L} = t^{1-N}\frac{d}{dt}\left(t^{N-1}\frac{d}{dt}\right).$$

In the past few years, in order to avoid transforming ramps into stairs (piecewise constant regions), several high-order PDEs were adopted [1–6]. And a number of authors hoped that these methods might perform better than some second-order equations. The problem (P) can be regarded as the analogue of the Euler-Lagrange equations from the variation problem in [1]. And the solution of the problem (P) can also be regarded as the steady-state case of the fourth-order anisotropic diffusion equation in [2–5]. As the radially symmetric form of the equation in (P), equation (1.1) has been the subject of intensive study in the recent decade [7–14]. Particularly, in [8], under some structure conditions, the authors obtained the existence of infinitely many positive symmetric radial entire solutions and investigated the asymptotic behavior of these solutions. From then on, more and more mathematical workers paid their attention to the problems on harmonic or polyharmonic equation(s). In [9, 10], Debnath and Xu studied the existence of infinitely many positive radically symmetric solutions for the singular nonlinear polyharmonic equation in $\mathbb{R}^2$ and $\mathbb{R}^N$ ($N \geq 3$), respectively. In [11, 12], Wu considered the nonlinear polyharmonic systems and equation of the type (1.1), respectively, and obtained some sufficient conditions for the existence of infinitely many radial positive entire solutions with the prescribed asymptotic behavior at infinity. In most of the works mentioned above, the authors used the operator $\Phi : C[0,\infty) \to C^2[0,\infty)$,

$$(\Phi h)(t) = \int_0^t s\ln\left(\frac{s}{t}\right)h(s)\,ds, \quad t \geq 0, \tag{1.3}$$

which was first represented in [15], and the fixed point theorem to obtain their main results. For $N = 2$, by a direct computation, one can easily see that

$$\mathscr{L}(\Phi h)(t) = h(t), \quad t \geq 0.$$

In this paper, we discuss the multiplicity of positive radially symmetric solutions for the problem (P), namely, problem (1.1)-(1.2). The main purpose of this paper is to investigate the existence, nonexistence and multiplicity of positive solutions of problem (1.1)-(1.2). Different from some known works, the equation we consider is quasilinear and it might have degeneracy or singularity. In fact, if $p > 2$, the equation is degenerate at the points where $\mathscr{L}u = 0$; while if $1 < p < 2$, then the equation has singularity at the points where $\mathscr{L}u = 0$. And for the restrictions of the boundary value conditions, the above-mentioned operator Φ in (1.3), which is suitable for the entire space, is inappropriate in this paper. Hence, we propose a new analogue of the operator Φ and apply the fixed point index theory combining with the upper and lower solutions method to investigate the multiplicity of positive radially symmetric solutions for the problem (P).

Assume that

(H1) $f : [0,+\infty) \to (0,+\infty)$ is continuous and nondecreasing on $[0,+\infty)$. Furthermore, there exist $\overline{\delta} > 0$ and $m > p-1$ such that $f(s) > \overline{\delta}s^m$, $s \in [0,+\infty)$;

(H2) $g : (0,1) \to (0,+\infty)$ is continuous, $0 < \int_0^1 s^{N-1}(s^{2-N}-1)g(s)\,ds < +\infty$, and $g(s) \not\equiv 0$ on any subinterval of $(0,1)$.

The main result of this paper is the following.

Theorem 1 *Let* (H1) *and* (H2) *hold true. Then there exists a threshold* $0 < \lambda^* < +\infty$ *such that problem* (1.1)-(1.2) *has no positive solution for* $\lambda > \lambda^*$, *has at least one positive solution for* $0 < \lambda \le \lambda^*$, *and especially has at least two positive solutions for* $0 < \lambda < \lambda^*$.

This paper is organized as follows. As preliminaries, we state some necessary lemmas in Section 2. In the last section, we apply the fixed point index theory and the upper and lower solutions method to prove the main result.

2 Preliminaries

In this section, we first present the necessary definitions and introduce some auxiliary lemmas, including those from the fixed point index theory and the theory based on the upper and lower solutions method.

Definition 1 Let $u(t) \in C^2[0,1] \cap C^4(0,1)$. We say that u is a positive solution of problem (1.1)-(1.2) if it satisfies (1.1)-(1.2) and $u(t) > 0$ on $(0,1)$.

Let $E = C[0,1]$ be a real Banach space with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$ and

$$P = \{u \in E : u(t) \ge 0, t \in [0,1]\}.$$

It is clear that P is a normal cone of E. Consider the following equation

$$\mathscr{T}x = x, \tag{2.1}$$

where $\mathscr{T} : E \to E$ is an operator defined on the Banach space E.

Definition 2 We say that u is an upper solution of equation (2.1) if it satisfies

$$\mathscr{T}u \le u.$$

If we change '$\le$' in the above inequality by '$\ge$', we can obtain the definition of a lower solution.

We say that u is a solution of problem (2.1), if u is an upper solution and is also a lower solution.

Throughout this paper, the main tools, which can be used to obtain the multiplicity of solutions for the problem, are the following two lemmas related to the fixed point index theory and the theory based on upper and lower solutions method, respectively, see [16].

Lemma 1 *Let* P *be a cone in a real Banach space* E, *let* Ω *be a bounded open subset of* E *with* $0 \in \Omega$, *and let* $A : P \cap \overline{\Omega} \to P$ *be a completely continuous operator. If*

$$Ax = \mu x, \quad x \in P \cap \partial\Omega \Rightarrow \mu \in (0,1),$$

then $i(A, P \cap \Omega, P) = 1$; *while if* $\inf_{x\in P\cap\partial\Omega} \|Ax\| > 0$ *and*

$$Ax = \mu x, \quad x \in P \cap \partial\Omega \Rightarrow \mu \notin (0,1],$$

then $i(A, P \cap \Omega, P) = 0$.

Lemma 2 *Suppose that X is a partially ordered Banach space, D is a normal subset of X, and $A : D \to X$ is a monotonic increasing complete continuous operator. If there exist x_0 and $y_0 \in D$ such that $x_0 \le y_0$, $\langle x_0, y_0\rangle \in D$, x_0, y_0 are the lower and the upper solution of the equation*

$$x - Ax = 0,$$

respectively, then the above equation has a minimum solution x^* and a maximum solution y^* in the ordered interval $\langle x_0, y_0\rangle$, and $x^* \le y^*$.

We also need the following technical lemma on the property of the function f, see [17].

Lemma 3 *Suppose that $f : [0,+\infty) \to (0,+\infty)$ is continuous. For $s > 0$ and $M > 0$, there exist $\bar{s} > s$ and $h_0 > 0$ such that*

$$sf(u+h) < \bar{s}f(u), \quad u \in [0,M], h \in (0,h_0).$$

In order to show the existence of the solutions, it is necessary to construct an appropriate operator and solve the corresponding operator equation. For this purpose, we notice that u is a solution of problem (1.1)-(1.2) if and only if u is a solution of the following problem:

$$\mathscr{L}v = \lambda g(t)f(u), \quad 0 < t < 1, \tag{2.2}$$

$$\mathscr{L}u = \varphi_q(v), \quad 0 < t < 1, \tag{2.3}$$

$$v'(0) = 0 = v(1), \tag{2.4}$$

$$u'(0) = 0 = u(1), \tag{2.5}$$

where $\varphi_q(s) = |s|^{q-2}s$, $1/p + 1/q = 1$. By (2.2) and (2.4), $v(t)$ can be expressed by

$$v(\tau) = -\lambda \int_\tau^1 \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} g(s)f\bigl(u(s)\bigr)\,ds \right) d\theta. \tag{2.6}$$

By (2.6) and the following formula of Dirichlet integral

$$\int_a^b \left(\int_a^x f(x,y)\,dy \right) dx = \int_a^b \left(\int_y^b f(x,y)\,dx \right) dy,$$

we have

$$\begin{aligned} v(\tau) &= -\lambda \int_\tau^1 \left(\int_0^\tau \left(\frac{s}{\theta}\right)^{N-1} g(s)f\bigl(u(s)\bigr)\,ds \right) d\theta - \lambda \int_\tau^1 \left(\int_\tau^\theta \left(\frac{s}{\theta}\right)^{N-1} g(s)f\bigl(u(s)\bigr)\,ds \right) d\theta \\ &= -\frac{\lambda}{N-2} \int_0^\tau s^{N-1}\bigl(\tau^{2-N} - 1\bigr) g(s)f\bigl(u(s)\bigr)\,ds \\ &\quad - \frac{\lambda}{N-2} \int_\tau^1 s^{N-1}\bigl(s^{2-N} - 1\bigr) g(s)f\bigl(u(s)\bigr)\,ds \\ &= -\lambda \int_0^1 k(\tau,s) g(s)f\bigl(u(s)\bigr)\,ds, \end{aligned}$$

where $k(\tau, s)$ is defined as follows:

$$k(\tau, s) = \begin{cases} \frac{s^{N-1}(\tau^{2-N}-1)}{N-2}, & 0 \le s \le \tau \le 1, \\ \frac{s^{N-1}(s^{2-N}-1)}{N-2}, & 0 \le \tau \le s \le 1. \end{cases}$$

It is easy to prove that $k(\tau, s)$ has the following properties.

Proposition 1 *For all* $\tau, s \in [0,1]$, *we have*

(i) $k(\tau, s) > 0$ *for* $(\tau, s) \in (0,1) \times (0,1)$;
(ii) $k(\tau, s) \le k(s,s) = \frac{s^{N-1}(s^{2-N}-1)}{N-2} = \frac{s-s^{N-1}}{N-2}$ *for* $0 \le \tau, s \le 1$;
(iii) $0 \le k(\tau, s) \le (N-2)(N-1)^{-(N-1)/(N-2)}$.

Proof We can easily obtain (i) and (ii) from the definition of $k(\tau, s)$. Now, we prove the conclusion (iii). Let $q(s) = k(s,s)$. We can see that

$$q(1) = 0, \qquad q(0) = \lim_{s \to 0^+} g(s) = 0,$$

and

$$q(s) > 0 \quad \text{for } 0 < s < 1.$$

So there exists s_0 such that $\max_{0 \le s \le 1} q(s) = g(s_0)$, where s_0 satisfies $g'(s_0) = 0$. Hence, $s_0 = (N-1)^{-1/(N-2)}$ and $g(s_0) = (N-2)(N-1)^{-(N-1)/(N-2)}$. The proof is complete. □

Then $u(t)$ can be expressed by

$$u(t) = \lambda^{\frac{1}{p-1}} \int_0^1 k(t, \tau) \varphi_q \left(\int_0^1 k(\tau, s) g(s) f\big(u(s)\big)\, ds \right) d\tau. \tag{2.7}$$

Next, we consider the following problem:

$$\mathscr{L}\big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big) = \lambda g(t) f(u), \quad t = |x|, 0 < t < 1, \tag{2.8}$$

$$u'(0) = 0, \qquad u(1) = h \ge 0, \qquad \big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big)'|_{t=0} = \big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big)|_{t=1} = 0. \tag{2.9}$$

We can define an integral operator $\mathscr{T}_\lambda^h : E \to E$, which is related to problem (2.8)-(2.9) by

$$\big(\mathscr{T}_\lambda^h u\big)(t) = h + \lambda^{\frac{1}{p-1}} \int_0^1 k(t, \tau) \varphi_q \left(\int_0^1 k(\tau, s) g(s) f\big(u(s)\big)\, ds \right) d\tau. \tag{2.10}$$

By (2.10), it is easy to obtain the following lemma, which is proved by a direct computation.

Lemma 4 *Let* (H1) *and* (H2) *hold true. Then problem* (1.1)-(1.2) *has a solution u if and only if u is a fixed point of* $\mathscr{T}_\lambda^0$. *And equations* (2.8)-(2.9) *have a solution u if and only if u is a fixed point of* $\mathscr{T}_\lambda^h$.

Now, we discuss the properties of the function $(\mathscr{T}_\lambda^h u)(t)$ for $u \in P$.

Lemma 5 *Let u be a positive solution of problem* (2.8)-(2.9). *Then*

$$\|u\| = \left(\mathscr{T}_\lambda^h u\right)(0),$$

i.e.,

$$\|u\| = h + \lambda^{\frac{1}{p-1}} \int_0^1 k(\tau,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)d\tau.$$

Proof By Lemma 4, we have

$$u_\lambda(t) = \left(\mathscr{T}_\lambda^h u\right)(t) = h + \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)d\tau.$$

Since

$$\begin{aligned}\left(\mathscr{T}_\lambda^h u\right)(t) = h + \lambda^{\frac{1}{p-1}}\Bigg(&\int_0^t \frac{\tau^{N-1}(t^{2-N}-1)}{N-2}\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)d\tau \\ &+ \int_t^1 \frac{\tau^{N-1}(\tau^{2-N}-1)}{N-2}\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)d\tau\Bigg),\end{aligned}$$

we have

$$u'(t) = \left(\mathscr{T}_\lambda^h u\right)'(t) = -\lambda^{\frac{1}{p-1}} \int_0^t \frac{\tau^{N-1}}{t^{N-1}}\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)d\tau \tag{2.11}$$

for any $t \in (0,1]$, and

$$u'(0) = -\lim_{t\to 0^+} \lambda^{\frac{1}{p-1}} t\varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u(s)\right)ds\right) = 0.$$

Consequently, $u_\lambda(t)$ is a strictly decreasing function on $[0,1]$. Hence

$$\left\|u(t)\right\| = \left(\mathscr{T}_\lambda^h u\right)(0) = h + \lambda^{\frac{1}{p-1}} \int_0^1 k(\tau,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)d\tau. \qquad \square$$

Lemma 6 *Assume $\{u_n\} \subset P$, $u \in P$, and $\|u_n - u\| \to 0$ as $n \to +\infty$. Then there exists $\theta_n \in (0,1)$ such that*

$$\left(\mathscr{T}_\lambda^h u_n\right)''(\theta_n) = 0 \quad \textit{and} \quad \left(\mathscr{T}_\lambda^h u_n\right)''(t) < 0 \quad \textit{for } t \in [0,\theta_n).$$

Furthermore,

$$\lim_{n\to\infty}\left(\mathscr{T}_\lambda^h u\right)''(\theta_n) = 0 \quad \textit{and} \quad \theta_n \to \theta,$$

where $\theta \in (0,1)$ and satisfies

$$\left(\mathscr{T}_\lambda^h u\right)''(\theta) = 0 \quad \textit{and} \quad \left(\mathscr{T}_\lambda^h u\right)''(t) < 0 \quad \textit{for } t \in [0,\theta).$$

Proof First, we want to prove that for any $u_n \in P$, there exists $\theta_n \in (0,1)$ such that $(\mathscr{T}_\lambda^h u_n)''(\theta_n) = 0$ and $(\mathscr{T}_\lambda^h u_n)''(t) < 0$ for $t \in [0,\theta_n)$.

From the proof of Lemma 5, we have

$$\left(\mathscr{T}_\lambda^h u_n\right)'(t) = -\lambda^{\frac{1}{p-1}} \int_0^t \frac{\tau^{N-1}}{t^{N-1}} \varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_n(s)\right) ds\right) d\tau. \tag{2.12}$$

And therefore the function $\mathscr{T}_\lambda^h u_n(t)$ is decreasing, and $\max_{t\in[0,1]}(\mathscr{T}_\lambda^h u_n)(t) = (\mathscr{T}_\lambda^h u_n)(0)$, and $(\mathscr{T}_\lambda^h u_n)'(0) = 0$. From (2.11), we have

$$\begin{aligned}\left(\mathscr{T}_\lambda^h u_n\right)''(t) = {}& \lambda^{\frac{1}{p-1}}(N-1) \int_0^t \frac{\tau^{N-1}}{t^N} \varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_n(s)\right) ds\right) d\tau \\ & - \lambda^{\frac{1}{p-1}} \varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u_n(s)\right) ds\right).\end{aligned} \tag{2.13}$$

From (2.13), using L'Hospital's rule, we can derive that for any $u_n \in P$,

$$\begin{aligned}\left(\mathscr{T}_\lambda^h u_n\right)''(0) &= \lim_{t\to 0^+} \left(\mathscr{T}_\lambda^0 u_n\right)''(t) \\ &= \lim_{t\to 0^+} \lambda^{\frac{1}{p-1}} \left((N-1)\int_0^t \frac{\tau^{N-1}}{t^N} \varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_n(s)\right) ds\right) d\tau \right. \\ &\quad \left. - \varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u_n(s)\right) ds\right)\right) \\ &= -\frac{1}{N}\varphi_q\left(\int_0^1 \frac{s^{N-1}(s^{2-N}-1)}{N-2} g(s)f\left(u_n(s)\right) ds\right) \\ &\le -\frac{1}{N}\varphi_q\left(\int_0^1 \frac{s^{N-1}(s^{2-N}-1)}{N-2} g(s)f(0)\, ds\right) < 0\end{aligned}$$

for any $u \in P$, which together with the following equation

$$\left(\mathscr{T}_\lambda^h u_n\right)''(1) = \lambda^{\frac{1}{p-1}}(N-1) \int_0^1 \tau^{N-1} \varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_n(s)\right) ds\right) d\tau > 0,$$

implies that for any $u_n \in P$, there exists $\theta_n \in (0,1)$ such that $(\mathscr{T}_\lambda^h u_n)''(\theta_n) = 0$ and $(\mathscr{T}_\lambda^h u_n)''(t) < 0$ for $t \in [0,\theta_n)$.

Next, we are going to prove the remainder of Lemma 6. By (2.13), we have Lemma 6. By (2.13), we have

$$\begin{aligned}&\left|\left(\mathscr{T}_\lambda^h u_n\right)''(t) - \left(\mathscr{T}_\lambda^h u\right)''(t)\right| \\ &\quad\le \lambda^{\frac{1}{p-1}}(N-1)\int_0^t \frac{\tau^{N-1}}{t^N}\left|\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_n(s)\right) ds\right)\right. \\ &\quad\quad \left. - \varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right) ds\right)\right| d\tau \\ &\quad\quad + \lambda^{\frac{1}{p-1}}\left|\varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u_n(s)\right) ds\right) - \varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u(s)\right) ds\right)\right| \\ &\quad\le \frac{2N-1}{N}\lambda^{\frac{1}{p-1}} \sup_{t\in[0,1]}\left|\varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u_n(s)\right) ds\right) - \varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u(s)\right) ds\right)\right|.\end{aligned}$$

We see that $u_n, u \in P$, which implies that $\|u\|$, $\|u_n\|$ are bounded respectively. And by $\|u_n - u\| \to 0$, we immediately infer that $\{u_n(t)\}$ are bounded uniformly. Then there must exist $M_0 > 0$ such that $|u_n(t)| < M_0$, $|u(t)| < M_0$ for any $t \in [0,1]$. Due to the continuity of $f(u)$, we see that $f(u_n)$ is continuous and bounded uniformly in $[0, M_0]$. Together with the continuity of $\varphi_q(s)$, (H1), (H2), and by using the Lebesgue dominated convergence theorem, it is not difficult to infer that $\|(\mathcal{T}_\lambda^h u_n)'' - (\mathcal{T}_\lambda^h u)''\| \to 0$. Hence, we have

$$\left|\left(\mathcal{T}_\lambda^h u\right)''(\theta_n)\right| = \left|\left(\mathcal{T}_\lambda^h u\right)''(\theta_n) - \left(\mathcal{T}_\lambda^h u_n\right)''(\theta_n)\right| \le \left\|\left(\mathcal{T}_\lambda^h u_n\right)'' - \left(\mathcal{T}_\lambda^h u\right)''\right\| \to 0.$$

By the front argumentation, $(\mathcal{T}_\lambda^h u)''(t) = 0$ has a null point $\theta \in (0,1)$ and $(\mathcal{T}_\lambda^h u)''(t) < 0$ for $t \in [0, \theta)$. If $\theta_n \nrightarrow \theta$, then there must exist a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \le \theta$ and $\theta_{n_k} \to \theta^* < \theta$. Due to $|(\mathcal{T}_\lambda^h u)''(\theta_n)| \to 0$, by the continuity of $(\mathcal{T}_\lambda^h u)''$, we then have $(\mathcal{T}_\lambda^h u)''(\theta^*) = 0$, which contradicts the fact that $(\mathcal{T}_\lambda^h u)''(\theta^*) < 0$. □

Now, we can define a cone $K \subset P$,

$$K = \left\{u \in P; \min_{t \in [\frac{1}{8}\theta, \frac{7}{8}\theta]} u(t) \ge 1/8\|u\|\right\},$$

where θ is defined in Lemma 6. It is clear that the nonnegative continuous concave functions are in K.

In order to apply the fixed point index theory, the following two lemmas, which relate to the monotonicity and the continuity of the operator $\mathcal{T}_\lambda^h$, are necessary. The proofs of Lemma 7 can be obtained immediately, by using (H1) and (2.10) with some simple direct computations.

Lemma 7 *Let* (H1) *and* (H2) *hold true. Then the operator* $\mathcal{T}_\lambda^h$ *defined by* (2.10) *is a monotonic increasing operator, i.e., if* $u_1(t) \le u_2(t)$, *then* $\mathcal{T}_\lambda^h u_1 \le \mathcal{T}_\lambda^h u_2$, *where* '$\le$' *is the partial order defined on* K.

Lemma 8 *Let* (H1) *and* (H2) *hold true. Then the operator* $\mathcal{T}_\lambda^h$ *is completely continuous, and* $\mathcal{T}_\lambda^h K \subset K$.

Proof Firstly, we testify the complete continuity of $\mathcal{T}_\lambda^h$. Let $\{u_n\} \subset K$, $u \in K$ with $\|u_n - u\| \to 0$ as $n \to +\infty$. Then we have

$$\begin{aligned}
&\left\|\left(\mathcal{T}_\lambda^h u_n\right)(t) - \left(\mathcal{T}_\lambda^h u\right)(t)\right\| \\
&\quad \le \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\left|\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_n(s)\right)ds\right) - \varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u(s)\right)ds\right)\right| d\tau \\
&\quad \le \lambda^{\frac{1}{p-1}} \sup_{t\in[0,1]}\left|\varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u_n(s)\right)ds\right) - \varphi_q\left(\int_0^1 k(t,s)g(s)f\left(u(s)\right)ds\right)\right|.
\end{aligned}$$

Since $\|u_n - u\| \to 0$ as $n \to \infty$ and $u_n, u \in K$, $\{u_n(t)\}$ is bounded uniformly, then there exists a constant $M_0 > 0$ such that $|u(t)| \le M_0$, $|u_n(t)| \le M_0$ for any $t \in [0,1]$, $n = 1, 2, \ldots$. Due to the continuity of $f(s)$, it follows that $f(u_n)$ is bounded uniformly in $[0, M_0]$. Moreover, because of the continuity of $\varphi_q(s)$, by the Lebesgue dominated convergence theorem and

(H2), we have

$$\|\mathscr{T}_\lambda^h u_n - \mathscr{T}_\lambda^h u\| \to 0 \quad (n \to \infty).$$

Therefore, we see that $\mathscr{T}_\lambda^h$ is continuous. And the compactness of the operator $\mathscr{T}_\lambda^h$ is easily obtained from the Arzela-Ascoli theorem.

Next, we want to obtain $\mathscr{T}_\lambda^h K \subset K$. For each $u \in K$, it is easy to check that $(\mathscr{T}_\lambda^h u)''(t) \le 0$ for $0 \le t \le \theta$; $(\mathscr{T}_\lambda^h u)(t) \ge 0$ for $0 < t < 1$ and $\max_{t\in[0,1]} \mathscr{T}_\lambda^h u(t) = \mathscr{T}_\lambda^h u(0)$. It follows from the concavity of $(\mathscr{T}_\lambda^h u)(t)$ on $[0,\theta]$ that each point on the chord between $(0, (\mathscr{T}_\lambda^h u)(0))$ and $(\theta, (\mathscr{T}_\lambda^h u)(\theta))$ is below the graph of $(\mathscr{T}_\lambda^h u)(t)$ on $[0,\theta]$. Thus,

$$\left(\mathscr{T}_\lambda^h u\right)(t) \ge \left(\mathscr{T}_\lambda^h u\right)(0) + \frac{(\mathscr{T}_\lambda^h u)(\theta) - (\mathscr{T}_\lambda^h u)(0)}{\theta - 0} t, \quad t \in \left[0, \frac{7}{8}\theta\right].$$

Hence,

$$\begin{aligned}
\min_{t\in[\frac{1}{8}\theta,\frac{7}{8}\theta]} \left(\mathscr{T}_\lambda^h u\right)(t) &\ge \min_{t\in[0,\frac{7}{8}\theta]} \left(\mathscr{T}_\lambda^h u\right)(t) \\
&\ge \min_{t\in[0,\frac{7}{8}\theta]} \left[\left(\mathscr{T}_\lambda^h u\right)(0) + \frac{(\mathscr{T}_\lambda^h u)(\theta) - (\mathscr{T}_\lambda^h u)(0)}{\theta - 0} t\right] \\
&= \min_{t\in[0,\frac{7}{8}\theta]} \left[\frac{(\theta - t)(\mathscr{T}_\lambda^h u)(0) + t(\mathscr{T}_\lambda^h u)(\theta)}{\theta}\right] \\
&\ge \frac{1}{8}\left(\mathscr{T}_\lambda^h u\right)(0) \\
&= \frac{1}{8}\|\mathscr{T}_\lambda^h u\|,
\end{aligned}$$

which implies $(\mathscr{T}_\lambda^h u)(t) \in K$. Hence, we obtain $\mathscr{T}_\lambda^h K \subset K$. □

Define

$$S = \left\{\lambda > 0; \text{such that problem (1.1)-(1.2) has at least one positive solution}\right\}.$$

Now, we give the *a priori* estimates on the positive solutions of problem (1.1)-(1.2).

Lemma 9 *Let* (H1) *and* (H2) *hold true. And suppose that* $\lambda \in S$, $S_1 = (\lambda, +\infty) \cap S \not\equiv \emptyset$. *Then there exists* $R(\lambda) > 0$ *such that* $\|u_{\lambda'}\| \le R(\lambda)$, *where* $\lambda' \in S_1$, *and* $u_{\lambda'} \in K$ *is a solution of problem* (1.1)-(1.2) *with* λ' *instead of* λ.

Proof For any fixed $\lambda' \in S$, let $u_{\lambda'}$ be a positive solution of problem (1.1)-(1.2). Then, by Lemma 4, we have

$$\begin{aligned}
u_{\lambda'}(t) &= \mathscr{T}_{\lambda'}^0 u_{\lambda'}(t) \\
&= \left(\lambda'\right)^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_{\lambda'}(s)\right)ds\right)d\tau.
\end{aligned}$$

Let

$$R(\lambda) = \max\left\{\left(\lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \times \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau\right)^{-\frac{p-1}{m-p+1}}, 1\right\}.$$

We claim that $\|u'_\lambda\| \le R(\lambda)$. Indeed, if $\|u_{\lambda'}\| < 1$, the result is easily obtained; while if $\|u_{\lambda'}\| \ge 1$, by (H1) and Lemma 5, we have

$$\begin{aligned}
\|u_{\lambda'}\| &= \left(\lambda'\right)^{\frac{1}{p-1}} \int_0^1 k(\tau,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\left(u_{\lambda'}(s)\right)ds\right)d\tau \\
&\ge \lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1}\|u_{\lambda'}\|^m \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau \\
&\ge \lambda^{\frac{1}{p-1}}\|u_{\lambda'}\|^{m/(p-1)} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|u_{\lambda'}\|^{m/(p-1)-1} \le \lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau, \\
&\|u_{\lambda'}\| \le \left(\lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \times \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau\right)^{-(p-1)/(m-p+1)}.
\end{aligned}$$

Therefore, $\|u_{\lambda'}\| \le R(\lambda)$. □

3 Existence of positive solutions

In this section, we give the proof of the main result, that is, Theorem 1. The proof will be divided into two parts. Firstly, by the upper and lower solutions method, we investigate the basic existence of positive solutions of problem (1.1)-(1.2). Exactly, we will determine the threshold λ^* of the parameter λ such that the problem is solvable if and only if $0 < \lambda \le \lambda^*$. Finally, by utilizing the fixed point index theory, we establish the multiplicity of positive solutions for the case $0 < \lambda < \lambda^*$.

We first present and prove the basic existence result of positive solutions of problem (1.1)-(1.2).

Proposition 2 *Let* (H1) *and* (H2) *hold true. Then there exists* $\lambda^* = \sup S$ *with* $0 < \lambda^* < +\infty$ *such that problem* (1.1)-(1.2) *admits at least one positive solution for* $\lambda \in (0, \lambda^*]$ *and has no positive solution for any* $\lambda > \lambda^*$.

Proof Let $\beta(t)$ be a solution of the following problem:

$$\begin{aligned}&\mathscr{L}\big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big) = g(t), \quad 0 < t < 1,\\ &u'(0) = u(1) = \big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big)'|_{t=0} = \big(|\mathscr{L}u|^{p-2}\mathscr{L}u\big)_{t=1} = 0.\end{aligned}$$

By (2.7), it has

$$\beta(t) = \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)\,ds\right)d\tau.$$

Let $\beta_0 = \max_{t\in[0,1]}\beta(t)$. Combining (H1) with (2.10), we obtain

$$\begin{aligned}\mathscr{T}_\lambda^0\beta(t) &\le \mathscr{T}_\lambda^0\beta_0\\ &= \lambda^{\frac{1}{p-1}}\int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(\beta_0)\,ds\right)d\tau\\ &\le \lambda^{\frac{1}{p-1}}f(\beta_0)^{\frac{1}{p-1}}\int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)\,ds\right)d\tau\\ &= \beta(t), \quad \text{for } 0 < \lambda < \frac{1}{f(\beta_0)},\end{aligned}$$

which implies that $\beta(t)$ is an upper solution of $\mathscr{T}_\lambda^0$. It is obvious that, for all $t \in [0,1]$, $\alpha(t) \equiv 0$ is a lower solution of $\mathscr{T}_\lambda^0$, and $\alpha(t) \le \beta(t)$, $t \in (0,1)$. Hence, $\mathscr{T}_\lambda^0 : \langle\alpha,\beta\rangle \to \langle\alpha,\beta\rangle$, where $\langle\alpha,\beta\rangle$ is the ordered interval in E. By Lemma 2, $\mathscr{T}_\lambda^0$ has a fixed point $u_\lambda \in \langle\alpha,\beta\rangle$ for $0 < \lambda < \frac{1}{f(\beta_0)}$. Therefore u_λ is a solution of problem (1.1)-(1.2). And then, for any $0 < \lambda < \frac{1}{f(\beta_0)}$, we have $\lambda \in S$, which implies that $S \neq \emptyset$.

On the other hand, if $\lambda_1 \in S$, then we must have $(0,\lambda_1) \subset S$. In fact, let u_{λ_1} be a solution of problem (1.1)-(1.2). Then, by Lemma 4, we have

$$u_{\lambda_1}(t) = \mathscr{T}_{\lambda_1}^0 u_{\lambda_1}(t), \quad t \in [0,1].$$

Therefore, for any $\lambda \in (0,\lambda_1)$, by (2.2), we have

$$\begin{aligned}\mathscr{T}_\lambda^0 u_{\lambda_1}(t) &= \lambda^{\frac{1}{p-1}}\int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\big(u_{\lambda_1}(s)\big)\,ds\right)d\tau\\ &\le \lambda_1^{\frac{1}{p-1}}\int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\big(u_{\lambda_1}(s)\big)\,ds\right)d\tau\\ &= \mathscr{T}_{\lambda_1}^0 u_{\lambda_1}(t)\\ &= u_{\lambda_1}(t),\end{aligned}$$

which implies that u_{λ_1} is an upper solution of $\mathscr{T}_\lambda^0$. Combining this with the fact that for $t \in [0,1]$, $\alpha(t) \equiv 0$ is a lower solution of $\mathscr{T}_\lambda^0$, and therefore, by Lemma 2, Lemma 4, Lemma 7 and Lemma 8, problem (1.1)-(1.2) has a solution, therefore $\lambda \in S$, which implies that $(0,\lambda_1) \subset S$.

Now, we claim that $\sup S < +\infty$. If this were not true, then we would have $\mathbb{N} \subset S$, where $\mathbb{N}$ denotes a natural number. Therefore, for any $n \in \mathbb{N}$, by the definition of S and Lemma 4, there exists $u_n \in K$ satisfying

$$u_n = \mathscr{T}_n^0 u_n = n^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\big(u_n(s)\big)\,ds\right)d\tau.$$

If $\|u_n\| \geq 1$, by Lemma 5, we have

$$\begin{aligned}
\|u_n\| &= n^{\frac{1}{p-1}} \int_0^1 k(\tau,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\big(u_n(s)\big)\,ds\right)d\tau \\
&\geq n^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1}\|u_n\|^m \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} \bar{\delta}k(s,s)g(s)\,ds\right)d\tau \\
&\geq \|u\|^{\frac{m}{p-1}} n^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\,ds\right)d\tau.
\end{aligned}$$

Consequently, we obtain

$$1 \geq n^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\,ds\right)d\tau. \tag{3.1}$$

If $\|u_n\| \leq 1$, by Lemma 5, we have

$$\begin{aligned}
1 &\geq \|u_n\| \\
&= n^{\frac{1}{p-1}} \int_0^1 k(\tau,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\big(u_n(s)\big)\,ds\right)d\tau \\
&\geq n^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)f\big(u_n(s)\big)\,ds\right)d\tau \\
&\geq n^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)f(0)\,ds\right)d\tau. \qquad (3.2)
\end{aligned}$$

Letting $n \to +\infty$ in (3.1)-(3.2), we can obtain the contradiction. Therefore we have $\sup S < +\infty$.

We are now in a position to determine the threshold λ^*. We conclude that

$$\lambda^* = \sup S.$$

It remains to show that $\lambda^* \in S$. Let$\{\lambda_n\} \subset [\frac{\lambda^*}{2}, \lambda^*)$, $\lambda_n \to \lambda^*$ $(n \to +\infty)$, $\{\lambda_n\}$ be an increasing sequence. Suppose that u_n is the solution of (1.1)-(1.2) with λ_n instead of λ. By Lemma 9, there exists $R(\frac{\lambda^*}{2}) > 0$ such that $\|u_n\| \leq R(\frac{\lambda^*}{2})$, $n = 1, 2, \ldots$. Therefore, $\{u_n\}$ is an equicontinuous and bounded uniformly subset in $C[0,1]$. By the Ascoli-Arzela theorem, $\{u_n\}$ has a convergent subsequence. Without loss of generality, we suppose $u_n \to u^*$ $(n \to +\infty)$. Since $u_n = \mathscr{T}_{\lambda_n}^0 u_n$, due to the continuity of $f(u)$, we see that $f(u_n)$ is continuous and bounded uniformly in $[0, R(\frac{\lambda^*}{2})]$, from which together with the continuity of $\varphi_q(s)$ and (H2), by the Lebesgue dominated convergence theorem, we have $u^* = \mathscr{T}_{\lambda^*}^0 u^*$. Hence, by Lemma 4, u^* is a solution of problem (1.1)-(1.2) with λ^* instead of λ. The proof is complete. □

Finally, we prove the main result in this paper.

Proof of Theorem 1 The arguments are based on fixed point index theory. Exactly speaking, we apply Lemma 1 to calculate the indexes of the corresponding operator in different two domains, and then complete the proof by the index theory.

Let K be the set defined in the previous section, namely

$$K = \left\{u \in P; \min_{t\in[\frac{1}{8}\theta,\frac{7}{8}\theta]} u(t) \geq 1/8\|u\|\right\}.$$

To calculate the index of the operator $\mathscr{T}_\lambda^0$ on some subset of K, we need to check the validity of the conditions in Lemma 1.

Let $\alpha(t) \equiv \bar{h}$ for $t \in [0,1]$. It is obvious that for any fixed $\lambda \in (0,\lambda^*)$, $\alpha(t)$ is a lower solution of the operator $\mathscr{T}_\lambda^{\bar{h}}$. On the other hand, by Lemma 9, there exists $R(\lambda) > 0$ such that $\|u_{\lambda'}\| \leq R(\lambda)$, where $\lambda' \in [\lambda,\lambda^*]$ and $u_{\lambda'}$ is a positive solution of problem (1.1)-(1.2) with λ' instead of λ. By Lemma 3, there exist $\overline{\lambda} \in (\lambda,\lambda^*)$ and $h_0 \in (0,1)$ satisfying

$$\lambda f(u+\bar{h}) < \overline{\lambda} f(u), \quad u \in \big[0,R(\lambda)\big], \bar{h} \in (0,h_0).$$

Let $u_{\overline{\lambda}}$ be a positive solution of problem (1.1)-(1.2) with $\overline{\lambda}$ instead of λ, and $\overline{u}_\lambda(t) = u_{\overline{\lambda}} + \bar{h}$, $\bar{h} \in (0,h_0)$. Then

$$\begin{aligned}
\overline{u}_\lambda(t) &= u_{\overline{\lambda}} + \bar{h} \\
&= \bar{h} + \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)\overline{\lambda} g(s) f\big(u_{\overline{\lambda}}(s)\big)\,ds\right)d\tau \\
&\geq \bar{h} + \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s) g(s) f\big(u_{\overline{\lambda}}(s)+\bar{h}\big)\,ds\right)d\tau \\
&= \mathscr{T}_\lambda^{\bar{h}}\overline{u}_\lambda(t),
\end{aligned}$$

which implies that $\overline{u}_\lambda(t)$ is an upper solution of the operator $\mathscr{T}_\lambda^{\bar{h}}$. Then, by Lemma 1, Lemma 4 and Lemma 8, problem (2.8)-(2.9) has a positive solution. Let $v_\lambda(t)$ be a solution of problem (2.8)-(2.9). Let $\Omega = \{u(t) < v_\lambda(t), t \in [0,1]\}$. It is clear that $\Omega \subset K$ is a bounded open set. If $u \in \partial\Omega$, then there exists $t_0 \in [0,1]$ such that $u(t_0) = v_\lambda(t_0)$. Therefore, for any $\mu \geq 1$, $h \in (0,h_0)$, $u \in \partial\Omega$, we have

$$\mathscr{T}_\lambda^0 u(t_0) < h + \mathscr{T}_\lambda^0 u(t_0) \leq h + \mathscr{T}_\lambda^0 v_\lambda(t_0) = \mathscr{T}_\lambda^h v_\lambda(t_0),$$

and by Lemma 9, it follows

$$\mathscr{T}_\lambda^h v_\lambda(t_0) = v_\lambda(t_0) = u(t_0) \leq \mu u(t_0).$$

Hence, for any $\mu \geq 1$, we have $\mathscr{T}_\lambda^0 u \neq \mu u$, $u \in \partial\Omega$. Therefore, Lemma 1 implies that

$$i\big(\mathscr{T}_\lambda^0, \Omega, K\big) = 1. \tag{3.3}$$

Now, we calculate the index of the operator $\mathscr{T}_\lambda^0$ on another relevant subset of K. For this purpose, we check the conditions of Lemma 1. Firstly, we check if condition (1) of Lemma 1 is fulfilled. In fact, for any $u \in K$, by (H1) and Lemma 5, we have

$$\begin{aligned}
\|\mathscr{T}_\lambda^0 u(t)\| &= \lambda^{\frac{1}{p-1}} \int_0^1 k(\tau,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f\big(u(s)\big)\,ds\right)d\tau \\
&\geq \lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1}\|u_\lambda\|^m \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau \\
&\geq \lambda^{\frac{1}{p-1}} \|u_\lambda\|^{m/(p-1)} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1}\right. \\
&\quad \times \left.\int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau \\
&\geq \|u\|^{\frac{m-p+1}{p-1}} \lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1}\right. \\
&\quad \times \left.\int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau\,\|u\|.
\end{aligned} \tag{3.4}$$

Choose $\overline{R} > 0$ such that

$$\overline{R}^{\frac{m-p+1}{p-1}} \lambda^{\frac{1}{p-1}} \int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(\tau,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^m \frac{(7\theta/8)^{2-N}-1}{(\theta/8)^{2-N}-1}\int_{\frac{1}{8}\theta}^{\frac{7}{8}\theta} k(s,s)g(s)\overline{\delta}\,ds\right)d\tau > 1.$$

Therefore, for any $R > \overline{R} > 0$ and $B_R \subset K$, by (3.4) we have

$$\|\mathscr{T}_\lambda^0 u\| > \|u\| > 0, \quad u \in \partial B_R, \tag{3.5}$$

where $B_R = \{u \in K \mid \|u\| < R\}$. If there exist $u_1 \in K \cap \partial B_R$ and $0 < \mu_1 \leq 1$ such that $\mathscr{T}_\lambda^0 u_1 = \mu_1 u_1$, then we can obtain $\|\mathscr{T}_\lambda^0 u_1\| \leq \|u_1\|$, which conflicts with (3.5). Therefore $\mu \notin (0,1]$. By using Lemma 1, we have

$$i(\mathscr{T}_\lambda^0, B_R, K) = 0. \tag{3.6}$$

Consequently, by the additivity of the fixed point index, we get

$$0 = i(\mathscr{T}_\lambda^0, B_R, K) = i(\mathscr{T}_\lambda^0, \Omega, K) + i(\mathscr{T}_\lambda^0, B_R\backslash\overline{\Omega}, K).$$

Since $i(\mathscr{T}_\lambda^0, \Omega, K) = 1$, $i(\mathscr{T}_\lambda^0, B_R\backslash\overline{\Omega}, K) = -1$. Therefore, there is a fixed point of $\mathscr{T}_\lambda^0$ in Ω and a fixed point of $\mathscr{T}_\lambda^0$ in $B_R\backslash\overline{\Omega}$, respectively. Finally, by utilizing Lemma 4, it follows that (1.1)-(1.2) has at least two positive solutions for the case $\lambda \in (0,\lambda^*)$, which combined with Proposition 2 yields Theorem 1. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SL and XH carried out the proof of the main part of this article, SL corrected the manuscript and participated in its design and coordination. All authors have read and approved the final manuscript.

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